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Computation of Hierarchical Renormalization-Group Fixed Points and Their ε-Expansions

K. Pinn,[1] A. Pordt,[1] and C. Wieczerkowski[1]

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We compute hierarchical renormalization-group fixed points as solutions to an algebraic equation for the coupling constants. This method does not rely on an iteration of renormalization-group transformations and therefore avoids the problem of fine tuning. We solve truncated versions of the fixed-point equation numerically for different values of the dimension parameter in the range $2<d<4$ and different orders of truncations. The method is well suited even for multicritical fixed points with any number of unstable directions. Precise numerical data are presented for the first three nontrivial fixed points and their critical indices. We also develop an ε-expansion for the hierarchical models using computer algebra. The numerical results are compared with the ε-expansion.

KEY WORDS: Renormalization group; renormalization-group fixed points; hierarchical model; epsilon-expansion; critical exponents.

1. INTRODUCTION

The renormalization group (RG)[1] is a nonperturbative approach to critical phenomena in statistical mechanics and Euclidean field theory. It offers a satisfactory explanation for the behavior of statistical models in the critical regime, in particular the appearance of universal quantities. It also suggests block spin transformations as an actual scheme for computations of critical quantities. Last but not least, it has proved to be a powerful tool in rigorous investigations. Although a lot of effort has been put into the subject since the seminal work of Wilson, it is far from being closed.

The main idea is to think of a critical system in terms of the flow of effective actions generated by a block spin transformation. This flow will have fixed points. An instructive example is a system with two possible

[1] Institut für Theoretische Physik I, Universität Münster, D-48149 Münster, Germany. Pinn@yukawa.uni-muenster.de.

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phases separated by a second-order transition in the absence of an external field. Then we typically find two stable fixed points characterizing the different phases and one unstable fixed point signaling a phase transition. Critical indices can be calculated as eigenvalues of a linearized transformation at the unstable fixed point. Given any critical system, the most important task is thus the determination of its RG flow fixed points and the flow in their vicinity.

Let us consider a Euclidean scalar field on a d-dimensional unit lattice Λ. The action is $S(\phi) = S_0(\phi) + V(\phi)$, with $V(-\phi) = V(\phi)$. The interaction is called local if the Boltzmann factor $\exp(-V(\phi))$ factorizes into a product $\prod_{x \in \Lambda} F(\phi(x))$. Unfortunately, locality is only preserved in an approximate sense by block spin transformations. A main technical difficulty therefore is to find an efficient parametrization of the effective actions generated by the flow. Also it is not *a priori* clear what the best choice for the block spin is. A guideline is to demand the effective actions to be approximately as local as possible.

The block spin transformation which we will consider here relies on a decomposition of the Gaussian measure $d\mu_v(\phi)$ [determined by the free action $S_0(\phi)$] into a background and a fluctuation part. The free covariance (propagator) is $v = (-\Delta)^{-1}$. The block spin transformation then consists of an integration step, a rescaling step of the background field and the block lattice, and a subsequent subtraction of the field-independent term. In this setup the nonlocality of the effective interaction is traced back to the fact that the fluctuation covariance, despite its exponential decay, couples fields further apart than a block distance.

In the hierarchical model[2] the free covariance $(-\Delta)^{-1}$ is replaced by a hierarchical counterpart $(-\Delta)^{-1}_{\mathrm{hier}}$. Both covariances share a similar long-distance decay, although the hierarchical covariance is not translation invariant, and the models therefore behave similarly in the critical region. As a gain, the hierarchical block spin transformation preserves locality in a strict sense. A hierarchical RG step turns out to be a nonlinear transformation of the local Boltzmann factor $F(\phi)$, which is a function of a single variable rather than a functional of a field configuration.

One point of view is to think of a hierarchical model as an approximation to its full brother. This is obviously only justified if there exists an interpolation between the two which admits the computation of systematic corrections around the hierarchical situation. Such a scheme is still lacking, although a number of first steps into this direction can be found in the literature (see, e.g., ref. 3). The main difference between the hierarchical and the full model is the absence of wave function renormalization in the former. Wave function renormalization comes about in the full model because the effective interaction contains a kinetic term as a part of the quadratic piece. This is clearly discarded once the effective action is forced to be ultralocal.

A second point of view is to think of hierarchical models as nontrivial statistical models by themselves which are perfectly suited for the RG approach. Due to their nontrivial interaction they are far from being exactly solvable. Their phase structure and critical behavior are as rich and complicated as in the full case. This point of view seems somewhat academic since we do not know of any real ensembles which belong to the universality class of a hierarchical model. Real systems do not show this peculiar breaking of translation invariance.

The third point of view is to consider hierarchical models as a test ground for both concepts and methods toward the RG treatment of the full model. The problem of nonlocalities in the effective action is believed to be of purely technical nature. This is convincingly supported by rigorous studies of many full critical models (e.g., refs. 4) which have been preceded by studies of their hierarchical counterparts. A full RG flow will always contain as a part a flow of local quantities which is accurately modeled by a hierarchical flow.

An exciting possibility within the hierarchical context is to tune the dimension parameter d away from integer values. The hierarchical transformation does not involve any lattice geometry and contains d and a scale parameter L as variables which can be taken real-valued. Let us stress that this can be done completely independent of perturbation theory. A beautiful picture emerges that shows a sequence of thresholds between two and four dimensions where new fixed points appear. For instance, four dimensions is the threshold below which an unstable double-well fixed point exists which governs three-dimensional physics. This picture is believed to be true also in the full model, although we do not know of an RG formulation which incorporates lattice geometry only in the form of a dimension parameter which can be tuned at will.

In this paper we investigate the question of RG flows in the hierarchical model. Usually small coupling constants are required to do analytically such an investigation. To study infrared fixed points we have no small coupling parameters at hand. Nevertheless, we will show that we can do RG calculations for models which are not asymptotically free, by using some algebraic equations.

After splitting off the quadratic fixed point, which corresponds to the unordered phase, we expand the Boltzmann factor in terms of local operators. We will use two kinds of operators, simple powers of fields and normal ordered products. Then the fixed-point equation becomes a set of algebraic equations for the coupling constants. We truncate this system and solve it numerically.

For the fact that truncation of the infinite system leads to correct results and how errors are estimated we refer to the work of Koch and Wittwer.[5] See also the more recent work in ref. 6. In ref. 5 it was shown by an exact β-function technique using certain norms that the infrared fixed point exists in three dimensions. In ref. 6, tools and methods are developed to control and estimate truncation errors.

The direct computation turns out to be much better than searching for fixed points by iterating block spin transformations and fine tuning the initial values. It allows us to determine even the higher fixed points with multidimensional unstable manifolds to high accuracy. Then we linearize this set of equations and diagonalize it numerically around the various fixed points. This directly computes the critical indices of the hierarchical model without having to rely on any kind of expansions.

The algebraic equations can also be written down for the full model in terms of a polymer expansion. It is less clear then how to truncate this system to a reasonable size. But we believe that this is manageable and will admit a similar numerical treatment.

The algebraic equations can be solved for any value of $d > 2$. Inserting a power series ansatz in ε for the coupling constants at $d = 4 - \varepsilon$, we obtain the ε-expansion for the double-well fixed point and analogously at the other critical dimensions. The resulting set of equations can be put into a recursive form which we iterate exactly using computer algebra. We compare this ε-expansion with the numerical results of the previous computations both for the fixed points and the critical indices.

Finally, we investigate the flow around the fixed points in terms of the eigenoperators of the linearized flow. It turns out that the eigenoperators at the double-well fixed point are approximately normal ordered powers in a small-field region but show exponential decay at large fields.

Hierarchical models have been investigated by many authors. Our approach is close to the constructive work of Koch and Wittwer on the nontrivial fixed point.[5] The bifurcation picture is also described within the frameork of a continuous hierarchical RG (as opposed to the lattice approach) in the article of Felder.[7] Another investigation of the ε-expansion for the double-well fixed point was done by Collet and Eckmann.[8]

The present contribution to the subject is an investigation of the flow on the computer in terms of an algebraic formulation for the fixed points, the critical indices, the eigenoperators, and the ε-expansion.

The paper is organized as follows. It consists of two parts. The hierarchical model is presented in Section 2. The algebraic equations (without normal ordering) are introduced and numerically solved in Section 3. In Section 4 we compute the critical indices as eigenvalues of the linearized transformation around the fixed points. The second part starts with Section 5. We present a concise derivation of a recursive ε-expansion as a power series solution to the algebraic equation in normal ordered form. The expansion is computed using computer algebra. Its results are compared with the numerical data.

2. THE MODEL

We consider hierarchical RG transformations, defined by

$$F \mapsto F', \qquad F'(\phi) = \int d\mu_\gamma(\zeta)\, F(\zeta + \beta\phi)^{L^d}$$
$$\beta = L^{1-d/2} \tag{1}$$

In Appendix A, we discuss the hierarchical model as an approximation of a field-theoretic model and motivate the hierarchical RG transformation.

Here $d\mu_\gamma$ denotes the one-dimensional Gaussian measure with covariance $\gamma > 0$, defined through

$$\int d\mu_\gamma(x)\, f(x) = (2\pi\gamma)^{-1/2} \int_{-\infty}^{+\infty} dx \exp\left(-\frac{x^2}{2\gamma}\right) f(x) \tag{2}$$

L denotes the block size of the corresponding blocking transformation. It is not difficult to convince oneself that

$$F_{\mathrm{HT}}(\phi) = L^{1/(L^d-1)} \exp\left(-\frac{L^2-1}{2\gamma L^d}\phi^2\right) \tag{3}$$

is a fixed point of the transformation (1). F_{HT} is called high-temperature fixed point. Let us write the Boltzmannian F as a product,

$$F(\phi) = F_{\mathrm{HT}}(\phi)\, Z(\phi) \tag{4}$$

The RG transformation of Z is again of the type (1),

$$Z \mapsto Z', \qquad Z'(\phi) = \int d\mu_{\gamma'}(\zeta)\, Z(\zeta + \beta'\phi)^{L^d} \tag{5}$$

with the changed parameters γ' and β' given by

$$\beta' = L^{-1-d/2}, \qquad \gamma' = L^{-2}\gamma \tag{6}$$

In the following, we shall always choose $L^d = 2$. As we shall see below, this choice will lead to quadratic fixed-point equations. Note, however, that it is not obvious how to realize a block spin transformation for such a scale factor on a lattice. For $L^d = 2$, we have

$$\beta = 2^{-(d-2)/(2d)}, \qquad \beta' = 2^{-(d+2)/(2d)} \tag{7}$$

Furthermore, we choose γ such that $\gamma' = \frac{1}{2}(1 - \beta'^2)$, i.e.,

$$\gamma = \tfrac{1}{2}(2^{2/d} - \tfrac{1}{2}) \tag{8}$$

This type of convention agrees with those of ref. 5. We study in this paper the fixed points of the RG transformation (5) that are even in ϕ, i.e., the even solutions of the fixed-point equation

$$Z(\phi) = \int d\mu_{\gamma'}(\zeta)\, Z(\zeta + \beta'\phi)^{L^d} \tag{9}$$

Let us remark that if Z is a solution of the fixed-point equation (9), then Z_δ, where $Z_\delta(\phi) := Z(\delta\phi)$, $\delta > 0$, is a solution of Eq. (9) if γ' is replaced by $\gamma'\delta^{-2}$. The consequence of dealing with Z instead of F is that β is replaced by $\beta' < \beta$. This has important consequences in the constructive approach of Koch and Wittwer.[5]

3. COMPUTATION OF NONTRIVIAL RG FIXED POINTS

A necessary ingredient for a successful study of the transformation (1) is a good choice of coordinates. Good here means that already a reasonable number of coupling constants provide a good approximation of the full problem. Furthermore, good coordinates should also be easy to deal with in practical calculations.

To study the transformation (1), one might consider the expansion of $F(\phi)$ in powers of ϕ. However, if such an expansion is truncated at a finite order, the resulting approximation will probably not define a reasonable (i.e., positive) Boltzmannian. The problem is that the coefficients change their sign from order to order, as in the Taylor expansion of $\exp(-x^2)$.

A good choice (though perhaps not the best) is to use the split equation (4) and consider a Taylor expansion of $Z(\phi)$ in powers of ϕ. This choice of coordinates has been successfully employed in a rigorous proof that the transformation (1) has a nontrivial fixed point in $d = 3$ dimensions.[5] It has the advantage that the expansion coefficients do not fluctuate in sign. Furthermore, the correct large-ϕ behavior of the Boltzmannian is implemented automatically.

3.1. Expansion of Z in Powers of ϕ^2

Let us define a rescaled function $P(\phi)$ through

$$Z(\phi) = P(\phi/(2\gamma')^{1/2}) \tag{10}$$

The RG transformation for P is

$$P \mapsto P', \qquad P'(\phi) = \int d\mu_{1/2}(\zeta)\, P(\zeta + \beta'\phi)^2 \tag{11}$$

We expand $P(\phi)$ in powers of ϕ^2,

$$P(\phi) = \sum_{l \geqslant 0} p_l \frac{\phi^{2l}}{2^l[(2l)!]^{1/2}} \tag{12}$$

The specific choice of normalization of the p_l in Eq. (12) turned out to be suitable for the numerical fixed-point solver to be described below. In terms of the expansion coefficients p_l, the RG reads

$$p_l \mapsto p'_l = \sum_{m,n} S_l^{mn} p_m p_n \tag{13}$$

with

$$S_l^{mn} = \begin{cases} \beta'^{2l} \left(\dfrac{1}{8}\right)^{-l+m+n} \dfrac{(2(m+n))!}{(m+n-l)!\,[(2l)!\,(2m)!\,(2n)!]^{1/2}} & \text{if } 0 \leqslant l \leqslant m+n \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

If we look for an RG fixed point, we have to study the infinite set of quadratic equations

$$0 = p_l^* - \sum_{m,n} S_l^{mn} p_m^* p_n^* \tag{15}$$

3.2. Numerical Solution of the Fixed-Point Equations

A straightforward numerical treatment of the problem defined by Eq. (15) becomes possible if we truncate the sum over l in Eq. (12), introducing a highest index $l_{\max}$,

$$P(\phi) = \sum_{l=0}^{l_{\max}} p_l \frac{\phi^{2l}}{2^l[(2l)!]^{1/2}} \tag{16}$$

Then, of course, also the sums over m and n run only from 0 to $l_{\max}$, and the fixed-point problem consists in the study of $l_{\max}+1$ quadratic equations,

$$\begin{aligned} &f_l = 0, \qquad 0 \leqslant l \leqslant l_{\max} \\ &f_l = p_l - \sum_{m,n \leqslant l_{\max}} S_l^{mn} p_m p_n \end{aligned} \tag{17}$$

We shall denote solutions of these equations by p_l^*. We used the routine C05NBF from the NAGLIB library[9] for a numerical solution of the fixed-point equations in the range $2<d<4$. Notice that the dimension dependence enters through the β' dependence of the structure coefficients S_l^{mn}. The program requires an initial guess of the solution. If the program is successful, it returns a solution, together with the values of the f_l for this particular solution. In Table I we show as an example the p_l^* for $l_{\max}=10$ and $d=3$, together with the f_l^* that can be considered as a measure of the error of the solution.

We used truncation parameters $l_{\max}$ in the range 10–50. For the two-well fixed points, $l_{\max}=20$ was completely sufficient. To get an impression of the finite-$l_{\max}$ effects, the reader is invited to study Table II, where we give the first 11 coefficients of the two-well fixed point in three dimensions for $l_{\max}=10$, 20, and 30.

We define an effective *potential* $V(\phi)$ through

$$V(\phi) := -\ln\left(\frac{F(\phi)}{F(\phi=0)}\right) \tag{18}$$

In Fig. 1 we show our results for the two-well potential for $d=2.1$ through $d=3.8$ in steps of 0.1. In all cases we used $l_{\max}=20$. The deepest potential

Table I. The p_l^* and f_l for $l_{\max}=10$ in $d=3$ Dimensions

l	p_l^*	f_l
0	$0.752806717034\times 10^{+0}$	$0.423398205312\times 10^{-16}$
1	$0.481272697982\times 10^{+0}$	$-0.113124959171\times 10^{-17}$
2	$0.313506765870\times 10^{+0}$	$0.347242613663\times 10^{-17}$
3	$0.186261032043\times 10^{+0}$	$-0.179714118083\times 10^{-17}$
4	$0.100696164171\times 10^{+0}$	$0.395226286632\times 10^{-17}$
5	$0.499270725225\times 10^{-1}$	$0.117550777471\times 10^{-16}$
6	$0.228929876623\times 10^{-1}$	$0.570517883570\times 10^{-17}$
7	$0.977563729148\times 10^{-2}$	$0.622745240266\times 10^{-17}$
8	$0.390718140134\times 10^{-2}$	$0.471372511658\times 10^{-17}$
9	$0.146546809430\times 10^{-2}$	$0.307351198867\times 10^{-17}$
10	$0.515497714660\times 10^{-3}$	$0.122533903732\times 10^{-17}$

Table II. The First 11 Coefficients p_l^* of the 2-Well in $d=3$ Dimensions for Three Different Values of l_{max}

l	$l_{max}=10$	$l_{max}=20$	$l_{max}=30$
0	$0.752806717034 \times 10^{-0}$	$0.752859732932 \times 10^{-0}$	$0.752859732933 \times 10^{-0}$
1	$0.481272697982 \times 10^{-0}$	$0.481191004612 \times 10^{-0}$	$0.481191004610 \times 10^{-0}$
2	$0.313506765870 \times 10^{-0}$	$0.313445974082 \times 10^{-0}$	$0.313445974081 \times 10^{-0}$
3	$0.186261032043 \times 10^{-0}$	$0.186254920100 \times 10^{-0}$	$0.186254920100 \times 10^{-0}$
4	$0.100696164171 \times 10^{-0}$	$0.100729195010 \times 10^{-0}$	$0.100729195011 \times 10^{-0}$
5	$0.499270725225 \times 10^{-1}$	$0.499755196476 \times 10^{-1}$	$0.499755196485 \times 10^{-1}$
6	$0.228929876623 \times 10^{-1}$	$0.229416956212 \times 10^{-1}$	$0.229416956222 \times 10^{-1}$
7	$0.977563729148 \times 10^{-2}$	$0.981866078064 \times 10^{-2}$	$0.981866078150 \times 10^{-2}$
8	$0.390718140134 \times 10^{-2}$	$0.394316100256 \times 10^{-2}$	$0.394316100333 \times 10^{-2}$
9	$0.146546809430 \times 10^{-2}$	$0.149410716099 \times 10^{-2}$	$0.149410716173 \times 10^{-2}$
10	$0.515497714660 \times 10^{-3}$	$0.536648340900 \times 10^{-3}$	$0.536648341727 \times 10^{-3}$

corresponds to $d=2.1$. With increasing dimension, the 2-well gets flatter and flatter until it vanishes in four dimensions.

From naive power counting and the studies of Section 5, one expects that n-well fixed points occur when the dimension goes below the threshold $d_n = 2n/(n-1)$. Note that exactly at these thresholds the operators ϕ^{2n}

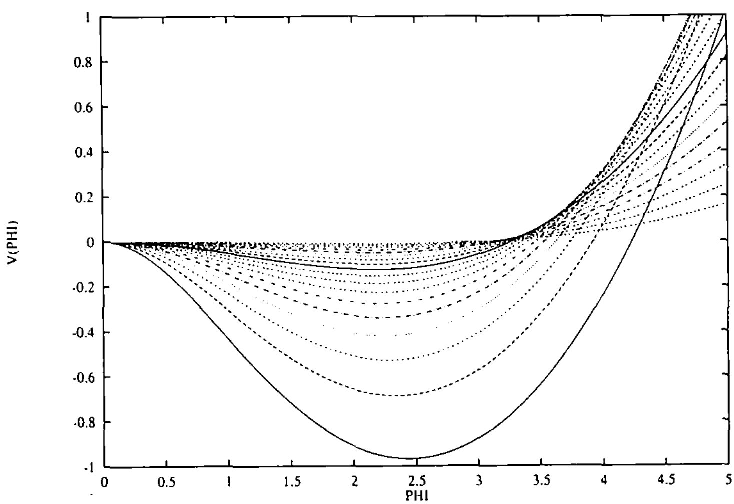

Fig. 1. Our results for the 2-well potential for $d=2.1$ through $d=3.8$ in steps of 0.1. The deepest potential corresponds to $d=2.1$. With increasing dimension, the 2-well gets flatter and flatter until it vanishes in four dimensions. In all cases we used $l_{max}=20$.

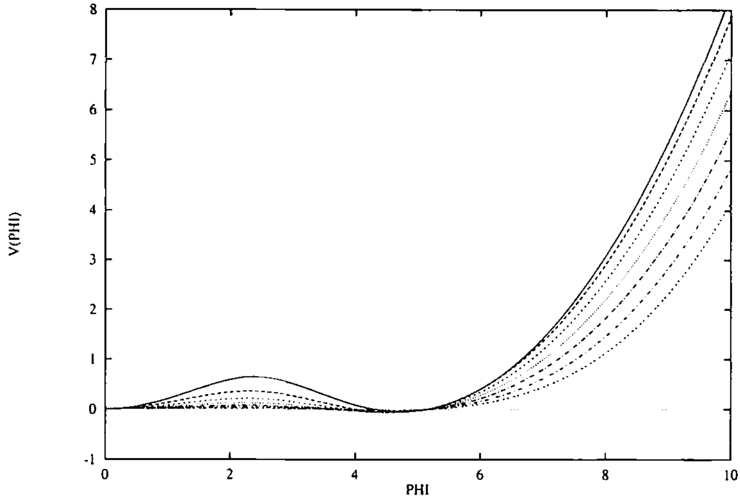

Fig. 2. The fixed-point 3-wells for $d = 2.1$ through $d = 2.7$ in steps of 0.1. The deepest potential corresponds to $d = 2.1$, the flattest one to $d = 2.7$. In all cases we used $l_{\max} = 20$.

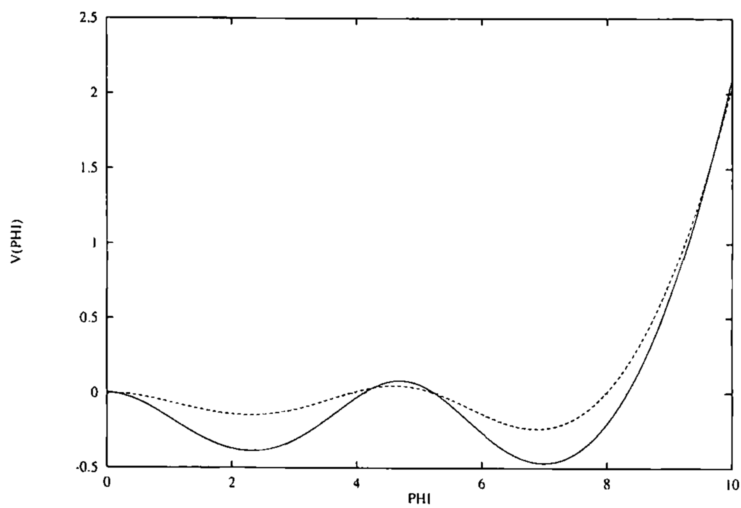

Fig. 3. The fixed-point 4-well potentials for $d = 2.1$ and $d = 2.3$. The potential with the deeper wells corresponds to $d = 2.1$. In both cases we used $l_{\max} = 30$.

become relevant with respect to the Gaussian fixed point. Thus 2-wells are expected to exist for $d<4$, 3-wells for $d<3$, 4-wells for $d<8/3$, and so on.

By starting our program with a suitable initial guess, we were able to find the three-well potentials. Figure 2 shows the results for $d=2.1$ through $d=2.7$ in steps of 0.1. The deepest potential corresponds to $d=2.1$, the flattest one to $d=2.7$. In all cases we used again $l_{\max}=20$.

The search for the four-well potential was a bit more difficult. It turned out that one needed more than 20 couplings for a reasonable parametrization. Figure 3 shows the four-well potential in $d=2.1$ and in $d=2.3$ dimensions. We used $l_{\max}=30$ in both cases.

In all cases, we made a further check on our results as follows: We represented the Boltzmannian F by its values on a grid of typically 300 sites on the interval $0\leqslant\phi\leqslant\Lambda$, where Λ was typically 10 or 20. The RG steps were then done by numerically performing the integral (1), using the NAGLIB routine D01GAF. This routine determines an approximation of an integral if the integrand is given on a finite number of points. All fixed points determined by the algebraic method were converted to functions on the ϕ-grid and then checked for stability under iterated application of the integration method.

4. THE LINEARIZED RENORMALIZATION GROUP

In this section we shall report on a numerical study of the eigenvalues of the hierarchical RG, linearized around the nontrivial fixed points. Note that the eigenvalues related to the full transformation (1) are the same as those related to the transformation (5) for Z. The reason is that fixed points F^* and Z^* differ only by the factor F_{HT} that is a fixed point itself. The transformation for Z around a fixed point parametrized by coordinates p_l^* is

$$(p_l^*+\delta_l)\mapsto(p_l^*+\delta_l')=\sum_{m,n} S_l^{mn}(p_m^*+\delta_m)(p_n^*+\delta_n) \tag{19}$$

We expand to first order in δ,

$$\delta_l\mapsto\delta_l'=\sum_n R_{ln}\delta_n \tag{20}$$

and identify the linearized RG transformation with the matrix R,

$$R_{ln}=2\sum_m S_l^{mn}p_m^* \tag{21}$$

The matrix R is not symmetric. It can, however, be shown that its eigenvalues are real (cf. Lemma 5.4 below). We used the NAGLIB procedures

F01AKF and F02APF to compute all eigenvalues λ_i of R, with $i = 0, ..., l_{max}$. All eigenvalues are positive in accordance with Corollary 2.3 of ref. 5. We define

$$\lambda_i = L^{a_i} \tag{22}$$

where L denotes the block size, in our case $2^{1/d}$.

Table III shows, as an example, the first six eigenvalues a_i of the linearized RG around the 2-well, the 3-well, and the 4-well in $d = 2.1$ dimensions, respectively. Positive eigenvalues a_i are called relevant, negative eigenvalues are called irrelevant, and a zero eigenvalue is called marginal. In accordance with the expectation, the n-wells have $n - 1$ (non-trivial) relevant eigenvalues.

One always observes an eigenvalue $a_0 = d$ that corresponds to the trivial volume operator. It corresponds to the fact that the fixed point Z^* itself is an eigenvector with eigenvalue $\lambda_0 = 2$. The next eigenvalue a_1 is related to the critical exponent ν, via[10]

$$\nu = \frac{1}{a_1} \tag{23}$$

In Table IV we show our results for the exponent ν for $d = 2.1$ through $d = 3.8$ in the case of a 2-well. It is interesting to compare the $d = 3$ result with the exponent ν for the three-dimensional Ising model. The best known estimates for the latter are in the range 0.624–0.630.[11] For $d \to 2$, the deviation of the exponent ν from the value in the full model increases. We know from the exact solution of the two-dimensional Ising model that $\nu = 1$ for $d = 2$. In the hierarchical model we observe $\nu > 2$ already at $d = 2.1$.

It is interesting to look at the eigenvectors of the linearized RG transformation. As an example we consider the eigenvectors of the transformation (1), linearized around the trivial fixed point $F^*(\phi) = 1$ and around

Table III. The Six Leading Eigenvalues a_i at $d = 2.1$ for the 2-Well, the 3-Well, and the 4-Well, Respectively

i	2-well	3-well	4-well
0	2.1000000	2.1000000	2.1000000
1	0.4787297	1.9715290	1.9851950
2	−1.1724335	0.4611953	1.0966180
3	−3.3445644	−0.7546772	0.4105210
4	−5.8303542	−2.2270214	−5.6565547
5	−8.4946711	−3.9790124	−1.7706071

Table IV. Results for the Critical Exponent ν in the Hierarchical Model for the 2-Well

d	ν	d	ν
2.1	2.08886	3.0	0.64957
2.2	1.36234	3.1	0.62570
2.3	1.09916	3.2	0.60484
2.4	0.95704	3.3	0.58640
2.5	0.86534	3.4	0.56995
2.6	0.79985	3.5	0.55516
2.7	0.74993	3.6	0.54182
2.8	0.71011	3.7	0.52973
2.9	0.67729	3.8	0.51877

the nontrivial two-well fixed point in $d=3$ dimensions. The eigenvectors $O_i^{(0)}(\phi)$ for the linearization around $F^*=1$ can be given exactly:

$$O_i^{(0)}(\phi) = \text{const} \cdot H_{2n}\left(\frac{\phi}{2\bar{\gamma}}\right) \tag{24}$$

with

$$\bar{\gamma} \equiv \frac{\gamma}{1-\beta^2} \tag{25}$$

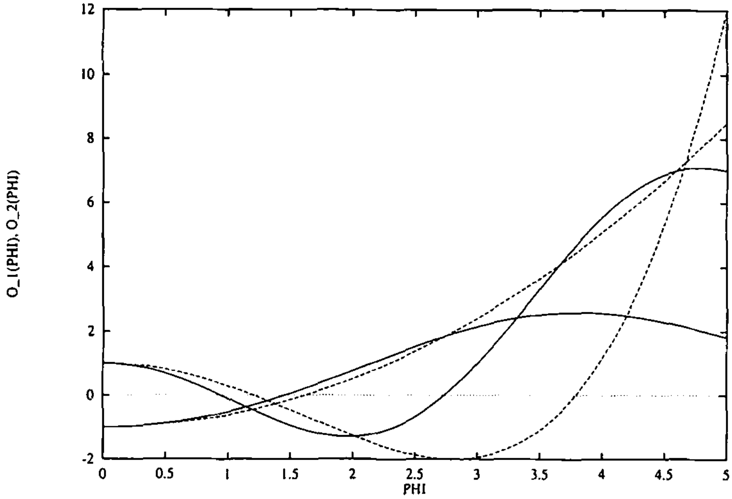

Fig. 4. Comparison of the two leading eigenvectors of the RG transformation linearized around the trivial fixed point and the 2-well fixed point in $d=3$. The single-well-shaped functions correspond to the relevant eigenvalue. The double-well-shaped functions correspond to the first irrelevant eigenvalue. The full lines belong to the nontrivial fixed point.

The H_k denote the Hermite polynomials. We compare these functions for $i = 1, 2$ with the corresponding eigenvectors of the RG linearized around the two-well fixed point in $d = 3$; see Fig. 4. The figure shows that the eigenvectors in $d = 3$ are of similar shape as the corresponding functions for the trivial fixed point. Note, however, that the "full" eigenvectors contain a factor F_{HT} and thus have a completely different large-ϕ behavior than the Hermite polynomials.

5. ϵ-EXPANSION

In this section, we want to compare the numerical results obtained in the previous section with expansions in $\varepsilon = d^* - d$, where d^* is one of the threshold dimensions $d_n = 2n/(n-1)$. This section is organized as follows: In Section 5.1 a parametrization of the hierarchical Boltzmannian is introduced that is suitable for the ε-expansion. The recursion relations for the expansion coefficients $a_l^{(k)}$ are derived in Section 5.2. In Section 5.3 we derive the recursion relations for the computation of the ε-expansion for the eigenvalues and eigenvectors of the linearized ε-expansion. In Section 5.4 we give some comparisons of the ε-expansion results with the numerical results.

In ref. 6 it is shown that the ε-expansion is locally Borel summable.

5.1. Expansion of F in Wick Monomials

We consider the expansion of the Boltzmannian F in terms of normal ordered powers of ϕ^2. This expansion shall be the basis for the ε-expansion to be studied below. One expands in terms of normal ordered monomials

$$F(\phi) = \sum_{l \geqslant 0} \frac{a_l}{\bar{\gamma}^l} :\phi^{2l}:_{\bar{\gamma}} \tag{26}$$

The reason to divide the coefficients a_l by $\bar{\gamma}^l$ is that a_l becomes $\bar{\gamma}^l$-independent for the fixed point. Under an RG step the coordinates a_l transform according to

$$a_l \mapsto a'_l = \beta^{2l} \sum_{m,n} \mathscr{C}_l^{mn} a_m a_n \tag{27}$$

The sum in Eq. (27) is restricted to $|m-n| \leqslant l \leqslant m+n$, and the "structure coefficients" are given by

$$\mathscr{C}_l^{mn} = \frac{(2m)!\,(2n)!}{(m+n-l)!\,(l+n-m)!\,(l+m-n)!} \tag{28}$$

Normal ordering with respect to a covariance $\bar{\gamma}$ is defined through

$$:\exp(a\phi):_{\bar{\gamma}} = \exp\left(-\frac{\bar{\gamma}}{2}a^2 + a\phi\right) \tag{29}$$

The normal ordered powers of ϕ can be expressed in terms of Hermite polynomials,

$$:\phi^n:_{\bar{\gamma}} = \left(\frac{\bar{\gamma}}{2}\right)^{n/2} H_n\left(\frac{\phi}{(2\bar{\gamma})^{1/2}}\right) \tag{30}$$

That we use a direct expansion of F as a basis for the ε-expansion seems to be in contradiction to our statement above that it should be much better to use the split $F = F_{\mathrm{HT}} Z$, and then expand Z. The ε-expansion is, however, an expansion about $F = 1$ (and *not* about $Z = 1$). Furthermore, as we shall see from the recursion relations to be derived below, the effective Boltzmannians to a given order in ε live in a finite-dimensional space of coupling constants. It is thus irrelevant which coordinates are chosen in this space as long as they span this space.

5.2. ϵ-Expansion for the a_l

In this section we derive the recursive relations for the coefficients $a_l^{(k)}$ defined through

$$a_l = \sum_{k=0}^{\infty} a_l^{(k)} \varepsilon^k \tag{31}$$

We determine by ε-expansion the coefficients of the infrared fixed point at $d = d^* - \varepsilon$ dimensions starting from the trivial fixed point in $d^* := 2l^*/(l^* - 1)$ dimensions, where $l^* \in \{2, 3, \ldots\}$. The following lemma shows how to compute the $a_l^{(k)}$ recursively.

Lemma 5.1. Suppose that the coefficients $(\beta^{-2l})^{(m)}$ are defined by

$$\beta^{-2l} = 2^{l/l^*} \sum_{m=0}^{\infty} (\beta^{-2l})^{(m)} \varepsilon^m \tag{32}$$

For $l^* \in \{2, 3, \ldots\}$ and $a_l^{(0)} = \delta_{l,0}$, we have

$$a_l^{(1)} = \alpha \delta_{l,l^*}, \qquad \alpha := \frac{2(\beta^{-2l^*})^{(1)}}{\mathscr{C}_{l^*}^{l^* l^*}} \tag{33}$$

and, for all $l \neq l^*$, $N \geqslant 1$,

$$a_l^{(N)} = \frac{1}{2^{l/l^*} - 2} \sum_{k=1}^{N-1} \left[-2^{l/l^*} (\beta^{-2l})^{(N-k)} a_l^{(k)} + \sum_{m,n} \mathscr{C}_l^{mn} a_m^{(N-k)} a_n^{(k)} \right] \quad (34)$$

For all $N \geqslant 2$,

$$a_{l^*}^{(N)} = -\frac{1}{2(\beta^{-2l^*})^{(1)}} \left\{ -2 \left[(\beta^{-2l^*})^{(N)} \alpha + \sum_{k=2}^{N-1} (\beta^{-2l^*})^{(N+1-k)} a_{l^*}^{(k)} \right] \right.$$
$$\left. + \sum_{k=2}^{N-1} \sum_{m,n} \mathscr{C}_{l^*}^{mn} a_m^{(N+1-k)} a_n^{(k)} + 2 \sum_{m:\, m \neq l^*} \mathscr{C}_{l^*}^{ml^*} a_m^{(N)} \alpha \right\} \quad (35)$$

Proof. The Nth-order term of the fixed-point equation

$$\beta^{-2l} a_l = \sum_{m,n} \mathscr{C}_l^{mn} a_m a_n \quad (36)$$

is given by

$$2^{l/l^*} \sum_{k=0}^{N} (\beta^{-2l})^{(N-k)} a_l^{(k)} = \sum_{k=0}^{N} \sum_{m,n} \mathscr{C}_l^{mn} a_m^{(N-k)} a_n^{(k)} \quad (37)$$

Since $a_l^{(0)} = \delta_{l,0}$ and $(\beta^{-0})^{(n)} = 0$ for $n > 0$, we obtain

$$2^{l/l^*} \left[a_l^{(N)} + \sum_{k=1}^{N-1} (\beta^{-2l})^{(N-k)} a_l^{(k)} \right] = 2a_l^{(N)} + \sum_{k=1}^{N-1} \sum_{m,n} \mathscr{C}_l^{mn} a_m^{(N-k)} a_n^{(k)} \quad (38)$$

We have used $\mathscr{C}_l^{m0} = \delta_{m,l}$. Equation (38) implies for $l \neq l^*$ Eq. (34). Consider Eq. (38) for the case $N = 1$:

$$2^{l/l^*} a_l^{(1)} = 2a_l^{(1)} \quad (39)$$

This implies $a_l^{(1)} = \alpha \delta_{l,l^*}$. To determine the constant α, we have to consider Eq. (38) for the case $N = 2$, $l = l^*$:

$$2(\beta^{-2l^*})^{(1)} a_{l^*}^{(1)} = \mathscr{C}_l^{l^*l^*} (a_{l^*}^{(1)})^2 \quad (40)$$

This implies

$$\alpha := \frac{2(\beta^{-2l^*})^{(1)}}{\mathscr{C}_{l^*}^{l^*l^*}} \quad (41)$$

Let us consider Eq. (38) for the case $N \geqslant 3$, $l = l^*$:

$$2 \sum_{k=1}^{N-1} (\beta^{-2l^*})^{(N-k)} a_{l^*}^{(k)} = \sum_{k=1}^{N-1} \sum_{m,n} \mathscr{C}_{l^*}^{mn} a_m^{(N-k)} a_n^{(k)} \quad (42)$$

This implies

$$2\left[(\beta^{-2l^*})^{(N-1)}\alpha+(\beta^{-2l^*})^{(1)}a_{l^*}^{(N-1)}+\sum_{k=2}^{N-2}(\beta^{-2l^*})^{(N-k)}a_{l^*}^{(k)}\right]$$
$$=2\sum_m \mathscr{C}_{l^*}^{ml^*}a_m^{(N-1)}\alpha+\sum_{k=2}^{N-2}\sum_{m,n}\mathscr{C}_{l^*}^{mn}a_m^{(N-k)}a_n^{(k)} \tag{43}$$

Thus,

$$a_{l^*}^{(N-1)}=\frac{1}{2((\beta^{-2l^*})^{(1)}-\mathscr{C}_{l^*}^{l^*l^*}\alpha)}\left\{-2\left[(\beta^{-2l^*})^{(1)}\alpha+\sum_{k=2}^{N-2}(\beta^{-2l^*})^{(N-k)}a_{l^*}^{(k)}\right]\right.$$
$$\left.+2\sum_{m:\,m\neq l^*}\mathscr{C}_{l^*}^{ml^*}a_m^{(N-1)}\alpha+\sum_{k=2}^{N-2}\sum_{m,n}\mathscr{C}_{l^*}^{mn}a_m^{(N-k)}a_n^{(k)}\right\} \tag{44}$$

Using $\mathscr{C}_{l^*}^{l^*l^*}\alpha=2(\beta^{-2l^*})^{(1)}$ and replacing N by $N+1$, we obtain Eq. (35). ∎

With the following lemma we provide explicit expressions for the ε-expansion of β^{-2l}.

Lemma 5.2. Suppose that the coefficients $(\beta^{-l})^{(m)}$ of the ε-expansion for the term β^{-l} are defined by

$$\beta^{-l}=2^{l/2l^*}\sum_{m=0}^{\infty}(\beta^{-l})^{(m)}\varepsilon^m \tag{45}$$

Then, we have $(\beta^{-l})^{(0)}=1$ and

$$(\beta^{-l})^{(m)}=\sum_{k=0}^{m-1}\frac{1}{(m-k)!}\left(-\frac{l\ln 2}{d^*}\right)^{m-k}\binom{m-1}{k}d^{*\,-m} \tag{46}$$

for $m\geqslant 1$.

Proof. We have

$$\beta^{-l}=2^{l/2l^*}2^{(l/d^*)(1-d^*/d)}=2^{l/2l^*}\exp\left[\frac{l\ln 2}{d^*}\left(1-\frac{1}{1-\varepsilon/d^*}\right)\right] \tag{47}$$

Expansion of the exponential function on the rhs of Eq. (47) gives

$$\beta^{-l}=2^{l/2l^*}\left[1+\sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{l\ln 2}{d^*}\right)^n\left(1-\frac{1}{1-\varepsilon/d}\right)^n\right] \tag{48}$$

Furthermore,

$$\left(1-\frac{1}{1-\varepsilon/d^*}\right)^n=\left(-\frac{\varepsilon}{d^*}\right)^n\sum_{k=0}^{\infty}\binom{n+k-1}{k}\left(\frac{\varepsilon}{d^*}\right)^k \tag{49}$$

Insertion of Eq. (49) into Eq. (48) yields

$$\beta^{-l} = 2^{l/2l^*} \left[1 + \sum_{n=1}^{\infty} \frac{1}{n!} \left(-\frac{l \ln 2}{d^*} \right)^n \sum_{k=0}^{\infty} \binom{n+k-1}{k} \left(\frac{\varepsilon}{d^*} \right)^{n+k} \right] \tag{50}$$

Introducing a new variable $m = n + k \in \{1, 2, \ldots\}$, we obtain

$$\beta^{-l} = 2^{l/2l^*} \left\{ 1 + \sum_{m=1}^{\infty} \left[\sum_{k=0}^{m-1} \frac{1}{(m-k)!} \left(-\frac{l \ln 2}{d^*} \right)^{m-k} \binom{m-1}{k} d^{*\,-m} \right] \varepsilon^m \right\} \tag{51}$$

This implies the assertion. ▌

5.3. ε-Expansion for the Linearized RG

We shall now consider the eigenvalue problem for the linearized RG equation. The eigenvalues and eigenvectors are computed by using the ε-expansion.

The linearized RG transformation is given by the matrix $U(a)$, with matrix elements

$$U_{nl}(a) := 2\beta^{2l} \sum_m \mathscr{C}_l^{mn} a_m \tag{52}$$

Lemma 5.3. Consider the eigenvalue equation for the linearized RG equation

$$U(a)\, b = \lambda b \tag{53}$$

Suppose that the vector a in $d = d^* - \varepsilon$ dimensions, $d^* = 2l^*/(l^* - 1)$, $l^* \in \{2, 3, \ldots\}$, is given by the ε-expansion

$$a_m = \sum_{k:\, k \geqslant 0} a_m^{(k)} \varepsilon^k \tag{54}$$

and the ε-expansion of U is

$$U = \sum_k U^{(k)} \varepsilon^k \tag{55}$$

Suppose that the ε-expansions of b and λ are given by

$$b = \sum_{m:\, m \geqslant 0} b^{(m)} \varepsilon^m, \qquad \lambda = \sum_{n:\, n \geqslant 0} \lambda^{(n)} \varepsilon^n \tag{56}$$

Suppose that $b_m^{(0)} = \delta_{n,\bar{m}}$ and $\lambda^{(0)} = 2^{1-\bar{m}/l^*}$, for $\bar{m} \in \{0, 1, 2, \ldots\}$. Then, we have

$$\lambda^{(N)} = \sum_{n=0}^{N-1} \sum_m U_{\bar{m}m}^{(N-n)} b_m^{(n)} \tag{57}$$

and, for $l \neq \bar{m}$,

$$b_l^{(N)} = \frac{1}{2(2^{-l/l^*} - 2^{-\bar{m}/l^*})} \sum_{n=0}^{N-1} \sum_m (\lambda^{(N-n)} \delta_{l,m} - U_{lm}^{(N-n)})\, b_m^{(n)} \tag{58}$$

Furthermore,

$$b_{\bar{m}}^{(N)} = 0 \tag{59}$$

The proof of this lemma will be given later.

Lemma 5.4. $U(a)$ is symmetric with respect to the canonical scalar product, defined by

$$(a, b) := \sum_n a_n b_n \tag{60}$$

Proof. We have

$$(u, U(a)\, v) = \sum_{n,l} u_n T_{nl} v_l \tag{61}$$

where

$$T_{nl} := \frac{\beta^{2n}}{(2n)!} U_{nl} \tag{62}$$

Since $T_{nl} = T_{ln}$, we have

$$(u, U(a)\, v) = (U(a)\, u, v) \quad \blacksquare \tag{63}$$

Lemma 5.4 shows that the eigenvalues of $U(a)$ are real. The next lemma shows how to compute the ε-expansion for the eigenvectors and eigenvalues.

Lemma 5.5. Suppose that the linearized RG group equation is given by the series expansion

$$U(a) = \sum_{n:\, n \geq 0} U^{(n)}(a)\, \varepsilon^n \tag{64}$$

and that the zeroth-order term $U^{(0)}$ is symmetric. Let b be an eigenvector with eigenvalue λ of $U(a)$, i.e.,

$$U(a)\, b = \lambda b \tag{65}$$

Let $b^{(0)}$ be a normalized eigenvector with eigenvalue $\lambda^{(0)}$ of $U^{(0)}(a)$, i.e.,

$$U^{(0)}(a)\, b^{(0)} = \lambda^{(0)} b^{(0)}, \qquad (b^{(0)}, b^{(0)}) = 1 \tag{66}$$

where the scalar product $(\cdot, \cdot)$ is defined by Eq. (60). Suppose that the eigenvalue $\lambda^{(0)}$ is not degenerate. Then, there exists an ε-expansion for the eigenvector b with eigenvalue λ

$$b = \sum_m b^{(m)} \varepsilon^m, \qquad \lambda = \sum_n \lambda^{(n)} \varepsilon^n \tag{67}$$

such that

$$(b^{(0)}, b^{(N)}) = 0 \tag{68}$$

for all $N > 0$. The coefficients $b^{(N)}$ and $\lambda^{(N)}$ are recursively determined by

$$\lambda^{(N)} = \sum_{n=0}^{N-1} (b^{(0)}, U^{(N-n)} b^{(n)}) \tag{69}$$

and

$$b^{(N)} = (U^{(0)} - \lambda^{(0)})^{-1} \left[\sum_{n=0}^{N-1} (U^{(N-n)} - \lambda^{(N-n)})\, b^{(n)} \right]^{\perp} \tag{70}$$

where $u^{\perp}$ is the component of u perpendicular to $b^{(0)}$. ▮

Proof. The eigenvalue equation implies, for all N,

$$\sum_{m,n:\, m+n=N} (U^{(m)} b^{(n)} - \lambda^{(m)} b^{(n)}) = 0 \tag{71}$$

Thus,

$$(U^{(0)} - \lambda^{(0)})\, b^{(N)} = \sum_{n=0}^{N-1} (U^{(N-n)} - \lambda^{(N-n)})\, b^{(n)} \tag{72}$$

Since $(U^{(0)} - \lambda^{(0)})\, b^{(0)} = 0$, we may add to $b^{(N)}$, $N > 0$, on the lhs of Eq. (72) a multiple of vector $b^{(0)}$ such that Eq. (88) holds. Scalar multiplication of Eq. (72) with $b^{(0)}$ gives

$$(b^{(0)}, (U^{(0)} - \lambda^{(0)})\, b^{(N)}) = \sum_{n=0}^{N-1} (b^{(0)}, (U^{(N-n)} - \lambda^{(N-n)})\, b^{(n)}) \tag{73}$$

Since $U^{(0)}$ is symmetric, the lhs of Eq. (73) is zero. Therefore Eq. (69) is valid. Since the rhs of Eq. (72) is perpendicular to $b^{(0)}$, Eq. (70) can be computed by Eq. (72). ∎

The next lemma presents the ε-expansion of the linearized RG transformation $U(a)$.

Lemma 5.6. Let the ε-expansion of β^{2l} be

$$\beta^{2l} = 2^{-l/l^*} \sum_{m=0}^{\infty} (\beta^{2l})^{(m)} \varepsilon^m, \qquad (\beta^{2l})^{(0)} = 1 \tag{74}$$

for $l^* \in \{2, 3, \ldots\}$. Then the ε-expansion of U

$$U = \sum_k U^{(k)} \varepsilon^k \tag{75}$$

is explicitly given by

$$U_{nl}^{(k)} = 2^{1-l/l^*} \sum_{r=0}^{k} \sum_m \mathscr{C}_l^{mn} (\beta^{2l})^{(k-r)} a_m^{(r)} \tag{76}$$

The proof of the foregoing lemma follows immediately from the definitions.

For the recursive computation of the coefficients for the eigenvalues and eigenvectors we need the start values of the recursion relations which are eigenvalues and eigenvectors of $U^{(0)}$.

Lemma 5.7. Suppose that $a_m^{(0)} = \delta_{m,0}$. The normalized eigenvectors $b^{(0)}$ with eigenvalues $\lambda^{(0)}$ of $U^{(0)}$ are

$$b_m^{(0)} = \delta_{m,\bar{m}}, \qquad \lambda^{(0)} = 2^{1-\bar{m}/l^*} \tag{77}$$

for all $\bar{m} \in \{0, 1, 2, \ldots\}$.

Proof. We have, using $a_m^{(0)} = \delta_{m,0}$, $\mathscr{C}_l^{0n} = \delta_{l,n}$,

$$U_{nl}^{(0)} = 2^{1-l/l^*} \delta_{n,l} \tag{78}$$

Thus

$$(U^{(0)} b^{(0)})_n = 2^{1-\bar{m}/l^*} \delta_{n,\bar{m}} = 2^{1-\bar{m}/l^*} b_n^{(0)} \quad ∎ \tag{79}$$

We finish this subsection with the proof of Lemma 5.3.

Proof of Lemma 5.3. By Lemma 5.5, Eq. (69), follows Eq. (57). Since

$$U_{nl}^{(0)} = 2^{1-l/l^*} \delta_{n,l}, \qquad \lambda^{(0)} = 2^{1-\bar{m}/l^*} \tag{80}$$

we have

$$(U^{(0)}-\lambda^{(0)})_{nl}=2(2^{-l/l^*}-2^{-\bar{m}/l^*})\,\delta_{n,l} \tag{81}$$

Therefore

$$(U^{(0)}-\lambda^{(0)})_{nl}^{-1}=\frac{1}{2(2^{-l/l^*}-2^{-\bar{m}/l^*})}\,\delta_{n,l} \tag{82}$$

Thus, Eq. (70) of Lemma 5.5 implies Eq. (58). ▮

5.4. Comparison of ε-Expansion with Numerical Results

We evaluated the recursion relations presented in the preceding subsection using the computer algebra program Maple V Release 2. This allowed us to go to relatively high order. We always used programs that computed everything exactly (in the form of analytical expressions) and programs that solved the recursion relations numerically. Note, however, that Maple allows for arbitrary high precision in the numerical computations. It was no problem to compute the coefficients $a_l^{(n)}$ for $d^*=4$ exactly to sixth order in ε. However, the expressions become quite nasty then. As an example we present the coefficients $a_l^{(n)}$ up to $n=3$ in Appendix B. The general structure for the expansion around $d^*=4$ is that at a given order

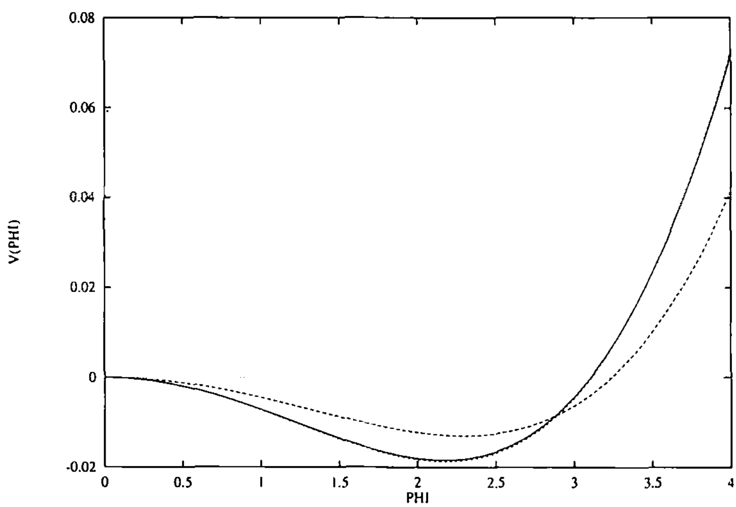

Fig. 5. Comparison of the "true" 2-well potential at $d=3.8$ with the first- and fourth-order ε-expansion. The full line gives the result obtained numerically, and the dashed lines give the first- and fourth-order approximations, respectively.

Table V. Numerical Results of a_1 and a_2 for n from 10 to 16

n	$a_1^{(n)}$	$a_2^{(n)}$
10	4.59314×10^1	-2.89444×10^2
11	-3.41664×10^2	2.26283×10^3
12	2.70284×10^3	-1.87364×10^4
13	-2.26138×10^4	1.63588×10^5
14	1.99286×10^5	-1.50104×10^6
15	-1.84404×10^6	1.44371×10^7
16	1.78725×10^7	-1.45243×10^8

n the only nonvanishing coefficients are those with $l \leqslant 2n$. The corresponding relation at $d^* = 3$ is $l \leqslant 3n$. The ε-expansion is expected to have zero convergence radius. However, the series are believed to be Borel-summable. For small ε even the naively summed low-order series can be a quite good approximation. In Fig. 5 we show the comparison of the "true" two-well potential at $d = 3.8$ with the first- and fourth-order ε-expansion. The full line gives the result obtained numerically, and the dashed lines give the first- and fourth-order approximations, respectively.

With the numerical version of the program it was no problem to go to orders like 16. In Table V we show the intimidating growth of the expansion coefficients when the order becomes large.

With the help of the recursion relations of the preceding subsections we also determined the ε-expansion for the exponent ν. We again used an exact version of the program that was practicable up to order 6, and a numerical version that could be used to higher order.

In Table VI we show our results for the expansion coefficients of ν, compared with those of the full model.[12] The first two orders are exactly

Table VI. Comparison of the ε-Expansion Coefficients of ν in the Hierarchical Model ν_i with the Ones in the Full Model $\nu_{i,f}$

i	ν_i	$\nu_{i,f}$
0	0.5000	0.5000
1	0.0833	0.0833
2	0.0556	0.0445
3	−0.0324	−0.0190
4	0.1468	0.0888
5	−0.5743	−0.2015

Table VII. Results of Resummed Series for ν up to Order ϵ^k for $k \leqslant 5$[a]

d	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	"true"	BP
3.0	0.6161	0.6523	0.6060	0.8255	0.3639	0.64957	—
3.1	0.6007	0.6293	0.5961	0.7228	0.4235	0.62570	0.62599
3.2	0.5864	0.6084	0.5855	0.6580	0.4771	0.60484	0.60136
3.3	0.5730	0.5895	0.5744	0.6144	0.5160	0.58640	0.58791
3.4	0.5605	0.5723	0.5630	0.5837	0.5369	0.56995	0.56973
3.5	0.5487	0.5568	0.5515	0.5612	0.5423	0.55516	0.55525
3.6	0.5377	0.5428	0.5402	0.5440	0.5379	0.54181	0.54183
3.7	0.5274	0.5302	0.5291	0.5303	0.5289	0.52973	0.52973
3.8	0.5177	0.5190	0.5186	0.5189	0.5187	0.51877	0.51877

[a] "true" is the numerical result, BP is the result of a Borel–Padé summation explained in the text.

the same (the coefficient ν_1 is 1/12 in the hierarchical and in the full model). This might be due to the fact that the ε-expansion of the exponent η starts at order ε^2.

In Table VII we give the results of resummed series for ν up to order ε^k, for $k \leqslant 5$, and $d = 3.0$–3.8. For the larger values of ε, the signal of the divergence of the series is obvious. For comparison we also quote our numerical result ("true") and the result of a Borel–Padé summation of the sixth-order ε-expansion (BP). The latter was obtained as follows: The ν_i were divided by $i!$, and the diagonal Padé approximation of the resulting Taylor series was determined. From the resulting rational function $Q(\varepsilon)$ the estimate for ν was then obtained by numerically computing

$$\nu(\varepsilon) = \int_0^\infty dt \exp(-t)\, Q(t\varepsilon) \tag{83}$$

There is a quite good agreement with the "true" results. (For $d = 3$, the diagonal Padé of the Borel transform had a nonintegrable singularity on the positive real axis.)

6. CONCLUSIONS

In this paper, we have computed and studied fixed points and exponents of hierarchical models. An extension to N-component models and general values of L could be easily done. Of course, many new ideas are necessary to do the same thing in full models. The crucial question here is the proper choice of parametrization of the Boltzmannian. An interesting problem which certainly deserves study is whether the ε-expansion for the

RG flow (in the Wilson sense) in full models could be performed to an order that is competitive with what has been done in the conventional framework.

APPENDIX A. FIELD THEORY AND HIERARCHICAL MODELS

This appendix discusses the hierarchical model as an approximation of a field-theoretic model and motivates the hierarchical RG equation.

The generating functional for Green's functions of a scalar field theory on a d-dimensional continuum $\mathbf{R}^d$ is given by the following (formal) infinite-dimensional integral:

$$Z[J] := \mathcal{N} \int \prod_{z \in \mathbf{R}^d} d\phi(z) \exp[-\tfrac{1}{2}(\phi, -\Delta\phi)] \exp[-V(\phi) + (J, \phi)] \tag{A1}$$

where ϕ is a scalar field and J is an external source on $\mathbf{R}^d$. Here $V(\phi) = \int_{z \in \mathbf{R}^d} \mathscr{V}(\phi(z))$ is the local interaction term and $\frac{1}{2}(\phi, -\Delta\phi)$ is the free part of the action. Δ is the Laplacian, and $\mathcal{N}$ is a normalization factor chosen such that

$$d\mu_{\mathrm{v}}(\phi) := \mathcal{N} \prod_{z \in \mathbf{R}^d} d\phi(z) \exp[-\tfrac{1}{2}(\phi, -\Delta\phi)] \tag{A2}$$

is a normalized Gaussian measure, v is called the propagator. The canonical scalar product is defined by

$$(\phi, \psi) := \int_{z \in \mathbf{R}^d} \phi(z)\, \psi(z) \tag{A3}$$

for fields ϕ, ψ on $\mathbf{R}^d$.

For RG calculations it is more convenient to use generating functionals with external fields ψ,

$$Z(\psi) := \int d\mu_{\mathrm{v}}(\phi) \exp[-V(\phi + \psi)] \tag{A4}$$

The two generating functionals are related by

$$Z(\psi) = Z[J] \exp[-\tfrac{1}{2}(J, \mathrm{v}J)]\,|_{J = \mathrm{v}^{-1}\psi} \tag{A5}$$

For the definition of the hierarchical model, we introduce the notion of a hierarchical lattice (or multigrid). For $L \in \{2, 3, ...\}$ and $j \in \mathbf{Z}$ divide $\mathbf{R}^d$ into hypercubes of side length $a_j := L^{-j}a$, where a is a unit length. Denote the set of all these hypercubes by Λ_j. We can consider Λ_j as a lattice with lattice spacing a_j by identifying the centers of the hypercubes with the lattice sites. The location of the hypercubes can be chosen in the following way. For a hypercube $y \in \Lambda_j$ let $\tilde{y}$ be the open hypercube of y. For $y \in \Lambda_j$ and $x \in \Lambda_k$ we suppose that $\tilde{y} \cap x = \varnothing$ or $\tilde{y} \subseteq x$. In the latter case we write $y \Subset x$. The hierarchical approximation is given by the replacement

$$\mathrm{v} \to \sum_{j \in \mathbf{Z}} \mathrm{v}^j \tag{A6}$$

where, for $z, z' \in \mathbf{R}^d$,

$$\mathrm{v}^j(z, z') := a_j^{2-d} \gamma \delta_{x,x'} \tag{A7}$$

Here, x and x' are the uniquely determined hypercubes of Λ_j such that $z \in x$, $z' \in x'$. We call v^j the fluctuation propagator. The above replacement represents the fact that in general v can be decomposed into a sum of fluctuation propagators which are exponentially decaying with decay length a_j. This exponential decay is simulated by the Kronecker delta on the rhs of Eq. (A7). There are other ways to define the hierarchical approximation. But all hierarchical approximations share the property that the fluctuation propagators v^j have compact support. Insertion of replacement (A6) into Eq. (A4) yields, using the convolution formula of Gaussian measures,

$$Z(\psi) = \int \prod_{j \in \mathbf{Z}} d\mu_{\mathrm{v}^j}(\phi^j) \exp\left[-V\left(\sum_{j \in \mathbf{Z}} \phi^j + \psi\right)\right] \tag{A8}$$

We define an ultraviolet cutoff by setting the propagator $\mathrm{v}^j = 0$ if $j > n$ and an infrared cutoff by setting $\mathrm{v}^k = 0$ if $k \leqslant j$:

$$Z_j^{(n)}(\psi) = \int \prod_{i=j+1}^{n} d\mu_{\mathrm{v}^i}(\phi^i) \exp\left[-V\left(\sum_{i=j+1}^{n} \phi^i + \psi\right)\right] \tag{A9}$$

The effective generating functions $Z_j^{(n)}$ and $Z_{j-1}^{(n)}$ are related by

$$Z_{j-1}^{(n)}(\psi) = \int d\mu_{\mathrm{v}^j}(\phi^j)\, Z_j^{(n)}(\phi^j + \psi) \tag{A10}$$

Since we started with a local interaction V, the effective generating functions obey the following factorization property:

$$Z_j^{(n)}(\psi) = \prod_{y:\, y \in \Lambda_j} Z_j^{(n)}(y \mid \psi) \tag{A11}$$

Since the kernel of v^j is constant on hypercubes of Λ_j, we can assume that the fields ψ are constant arguments of $Z_j^{(n)}$, i.e., do not depend on $z \in \mathbf{R}^d$. Therefore, Eq. (A10) is equivalent to

$$Z_{j-1}^{(n)}(x \mid \psi) = \prod_{y:\, y \in x} \left[\int d\mu_{v^j}(\phi)\, Z_j^{(n)}(y \mid \phi + \psi) \right] \tag{A12}$$

for $x \in \Lambda_{j-1}$. For translation-invariant models $Z_j^{(n)}(y \mid \phi)$ does not depend on $y \in \Lambda_j$. Let us define

$$Z_j^{(n)}(\phi) := Z_j^{(n)}(y \mid a_j^{1-d/2} \phi) \tag{A13}$$

From Eq. (A12) and definition Eq. (A13) follows, using $\psi \to a_{j-1}^{1-d/2} \psi$,

$$Z_{j-1}^{(n)}(\psi) = \left[\int d\mu_\gamma(\phi)\, Z_j^{(n)}(\phi + L^{1-d/2} \psi) \right]^{L^d} \tag{A14}$$

For $F' := (Z_{j-1}^{(n)})^{L^{-d}}$ and $F := (Z_j^{(n)})^{L^{-d}}$, we obtain the hierarchical RG transformation (1).

APPENDIX B. SOME RESULTS OF ϵ-EXPANSION FOR THE $a_i^{(n)}$

$$\begin{aligned} R &:= \ln 2 \\ T &:= \sqrt{2} \end{aligned} \tag{B1}$$

$$a_0^{(0)} = 1$$

$$a_0^{(2)} = -\frac{R^2}{864}$$

$$a_0^{(3)} = \frac{12R^2(3R-2) - R^2(19R-18)\,T}{-31104T + 41472}$$

$$a_1^{(2)} = \frac{R^2}{216T - 432}$$

$$a_1^{(3)} = \frac{-2R^2(7R-8) + R^2(7R-12)\,T}{17280T - 24192}$$

$$a_2^{(1)} = -\frac{R}{144}$$

$$a_2^{(2)} = \frac{12R(3R-2) - R(19R-18)\,T}{-10368T + 13824}$$

$$a_2^{(3)} = \frac{R(154R^2+1161R-459)-12R(8R^2+69R-27)\,T}{-746496T+1057536}$$

$$a_3^{(2)} = \frac{R^2}{2592T-2592}$$

$$a_3^{(3)} = \frac{-8R^2(R-1)+R^2(5R-6)\,T}{72576T-103680}$$

$$a_4^{(2)} = \frac{R^2}{41472}$$

$$a_4^{(3)} = \frac{-12R^2(3R-2)+R^2(35R-18)\,T}{-1492992T+1990656}$$

$$a_5^{(3)} = -\frac{R^3}{373248T-373248}$$

$$a_6^{(3)} = -\frac{R^3}{17915904} \tag{B2}$$

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